

Extended supersymmetry of semichiral sigma model in $4D$.

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Abstract

Briefly: Using a novel $(1,1)$ superspace formulation of semichiral sigma models with $4D$ target space, we investigate if an extended supersymmetry in terms of semichirals is compatible with having a $4D$ target space with torsion.

In more detail: Semichiral sigma models have $(2,2)$ supersymmetry and Generalized Kähler target space geometry by construction. They can also support $(4,4)$ supersymmetry and Generalized Hyperkähler geometry, but when the target space is four dimensional indications are that the geometry is restricted to Hyperkähler. To investigate this further, we reduce the model to $(1,1)$ superspace and construct the extra (on-shell) supersymmetries there. We then find the conditions for a lift to $(2,2)$ super space and semichiral fields to exist. Those conditions are shown to hold for Hyperkähler geometries. The $SU(2) \otimes U(1)$ WZW model, which has $(4,4)$ supersymmetry and a semichiral description, is also investigated. The additional supersymmetries are found in $(1,1)$ superspace but shown *not* to be liftable to a $(2,2)$ semichiral formulation.

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1 Introduction

Generalized Kähler geometry is efficiently probed by $(2,2)$ supersymmetric sigma models in $D = 2$, [1]. Of particular interest for the present investigation is the symplectic case, i.e., sigma models that depend on semichiral superfields only. Additional supersymmetries for these models were discussed in [2], and in [3]. In the latter article focus is on four-dimensional target spaces and it is shown that a very general ansatz for additional supersymmetries leads to an on-shell extended supersymmetry and restricts the target space geometry to be hyperkähler.

In [3] this is seen as a shortcoming of the ansatz, since it is argued that the $SU(2) \otimes U(1)$ WZW model of [4] constitutes a counterexample. It has nonzero torsion and when coordinatized by chiral and twisted chiral superfields it has “manifest” $(4, 4)$ supersymmetry. It further has a dual semichiral description [5] which is then expected to also display the $(4, 4)$ supersymmetry¹.

In this paper we investigate the possibility that the $(2, 2)$ semichiral conditions are incompatible with “manifest” $(4, 4)$ transformations.² To study this problem, we descend to $(1, 1)$ superspace and develop an on-shell formalism for the extra super symmetries, a formulation which retains the relation to $(2, 2)$ semichirals. We test this $(1, 1)$ formalism on the second supersymmetry (which is non-manifest in $(1, 1)$) and then apply it to a hyperkähler geometry which is shown to satisfy the conditions for having a $(2, 2)$ semichiral realisation, as expected from [3].

We also derive the extra supersymmetries for the WZW model [4] in $(1, 1)$ superspace in the relevant coordinates. When subjected to the same test they fail to satisfy some of the conditions. This leads to the surprising conclusion that $(4, 4)$ supersymmetry in a $(1, 1)$ formulation of a $(2, 2)$ sigma model with on-shell supersymmetry is incompatible with the introduction of the $(2, 2)$ auxiliary fields.

2 Background

2.1 Semichiral sigma models

Consider a generalized Kähler potential [1] with one left- and one right semichiral field and their complex conjugates, $K(\mathbb{X}^L, \mathbb{X}^R)$, where $L = (\ell, \bar{\ell})$ and $R = (r, \bar{r})$. The action,

$$S = \int d^2x d^2\theta d^2\bar{\theta} K(\mathbb{X}^L, \mathbb{X}^R) \quad (2.1)$$

has manifest $N = (2, 2)$ supersymmetry. The supersymmetry algebra is defined in terms of the anti-commutator of the covariant supersymmetry derivatives as

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial_\pm \quad (2.2)$$

and the semichiral fields are defined by their chirality constraints as [7]

$$\bar{\mathbb{D}}_+ \mathbb{X}^\ell = 0, \quad \bar{\mathbb{D}}_- \mathbb{X}^r = 0. \quad (2.3)$$

¹By “manifest” we shall mean “as realised by transformations of $(2, 2)$ superfields.”

²Another case of supersymmetries being obstructed occurs when dualisation is along isometries that do not commute with the extra super symmetries. This leads to nonlocal realisations of the extra susys in the dual model [6]. Here, however, the extra susys commute with the isometry used in dualising.

The geometry of the model is bi-hermitean [8], [7], governed by two complex structures $J^{(+)}$ and $J^{(-)}$ that both preserve the metric \mathbb{G}

$$J^{(\pm)t} \mathbb{G} J^{(\pm)} = \mathbb{G} \quad (2.4)$$

as well as by an anti-symmetric B -field whose field strength H enters in the form of torsion in the covariant constancy conditions

$$0 = \nabla^{(\pm)} J^{(\pm)} = \left(\partial + \Gamma^{(0)} \pm \frac{1}{2} H \mathbb{G}^{-1} \right) J^{(\pm)} , \quad (2.5)$$

where $\Gamma^{(0)}$ is the Levi-Civita connection. These conditions identify the geometry as bi-hermitean [8], or generalized Kähler geometry (GKG) [9].

The fact that our superfields are semichiral specifies the GKG as being of symplectic type where the metric g and the \mathbb{B} -field take the form³

$$\begin{aligned} \mathbb{G} &= \Omega[J^{(+)}, J^{(-)}] \\ \mathbb{B} &= \Omega\{J^{(+)}, J^{(-)}\} . \end{aligned} \quad (2.6)$$

The matrix Ω is defined as

$$\Omega = \frac{1}{2} \begin{pmatrix} 0 & K_{LR} \\ -K_{RL} & 0 \end{pmatrix} \quad (2.7)$$

and the submatrix K_{LR} is the Hessian

$$K_{LR} = \begin{pmatrix} K_{\ell r} & K_{\ell \bar{r}} \\ K_{\bar{\ell} r} & K_{\bar{\ell} \bar{r}} \end{pmatrix} . \quad (2.8)$$

An additional condition results from the target space being four-dimensional and reads [7]

$$\{J^{(+)}, J^{(-)}\} = 2c, \quad \Rightarrow \mathbb{B} = 2c \Omega , \quad (2.9)$$

where c in general is a function of the coordinates.

2.2 Extra SUSY

In [3] it is shown that a general ansatz for $(4, 4)$ susy in a semichiral sigma model

$$\begin{aligned} \delta \mathbb{X}^\ell &= \epsilon^+ \bar{\mathbb{D}}_+ f(\mathbb{X}^L, \mathbb{X}^R) + g(\mathbb{X}^\ell) \epsilon^- \bar{\mathbb{D}}_- \mathbb{X}^\ell + h(\mathbb{X}^\ell) \epsilon^- \mathbb{D}_- \mathbb{X}^\ell , \\ \delta \mathbb{X}^{\bar{\ell}} &= \epsilon^+ \mathbb{D}_+ \bar{f}(\mathbb{X}^L, \mathbb{X}^R) + \bar{g}(\mathbb{X}^{\bar{\ell}}) \epsilon^- \mathbb{D}_- \mathbb{X}^{\bar{\ell}} + \bar{h}(\mathbb{X}^{\bar{\ell}}) \epsilon^- \bar{\mathbb{D}}_- \mathbb{X}^{\bar{\ell}} , \\ \delta \mathbb{X}^r &= \epsilon^- \bar{\mathbb{D}}_- \tilde{f}(\mathbb{X}^L, \mathbb{X}^R) + \tilde{g}(\mathbb{X}^r) \epsilon^+ \bar{\mathbb{D}}_+ \mathbb{X}^r + \tilde{h}(\mathbb{X}^r) \epsilon^+ \mathbb{D}_+ \mathbb{X}^r , \\ \delta \mathbb{X}^{\bar{r}} &= \epsilon^- \mathbb{D}_+ \tilde{\bar{f}}(\mathbb{X}^L, \mathbb{X}^R) + \tilde{\bar{g}}(\mathbb{X}^{\bar{r}}) \epsilon^+ \mathbb{D}_+ \mathbb{X}^{\bar{r}} + \tilde{\bar{h}}(\mathbb{X}^{\bar{r}}) \epsilon^+ \bar{\mathbb{D}}_+ \mathbb{X}^{\bar{r}} , \end{aligned} \quad (2.10)$$

³This gives the \mathbb{B} field in a particular global gauge as $\mathbb{B} = \mathbb{B}^{(2,0)} + \mathbb{B}^{(0,2)}$ with respect to both complex structures.

leads to invariance and closure of the algebra only on-shell and provided that the geometry is hyperkähler. The on-shell requirement follows from

$$[\delta_1, \delta_2]\ell = -\epsilon_{[2}^+\epsilon_{1]}^+|f_\ell|^2\partial_{++}\dots \quad (2.11)$$

which has the wrong sign for supersymmetry. It is an interesting fact that on-shell closure of the algebra, together with conditions that come from invariance of the action, requires that the function $c(\mathbb{X}^L, \mathbb{X}^R)$ defined by (2.11) is constant with absolute value less than one, which means that the geometry is hyperkähler. This on-shell closure is different than the one which arises in the general $(1, 1)$ discussion of extended susy [8] which locates the non-closure of the algebra to the $(+, -)$ sector where the commutator $[J^{(+)}, J^{(-)}]$ multiplies the field equation. In the present case left or right susy alone require field equations.

In [3] we argue that the ansatz for the additional supersymmetry is too restrictive and should include central charge transformations. The reason for trying to find a more general ansatz is that there is a known example of a BiLP⁴ with $(4, 4)$ supersymmetry, mentioned in the introduction, that has a dual semichiral formulation which manifestly violates the hyperkähler condition [5]. The duality and extra supersymmetry will be discussed in detail in Sec.4 below. Since the isometry used in the dualisation commutes with the extra supersymmetry, the dual model is expected to have the extra symmetry as well. To investigate this, bearing in mind that the relevant algebra only closes on-shell, we now develop a novel $N=(1, 1)$ form of the semichiral model and its additional symmetries.

3 $N=(4, 4)$ in $N=(1, 1)$ superspace

We want to find out under what conditions a semichiral sigma model in $4D$ supports additional complex structures forming an $SU(2)$ algebra⁵

$$I_{(+)}^{(\mathfrak{a})}I_{(+)}^{(\mathfrak{b})} = -\delta^{\mathfrak{ab}} + \epsilon^{\mathfrak{abc}}I_{(+)}^{(\mathfrak{c})} \quad (3.12)$$

with $\mathfrak{a} = 1, 2, 3$ and the identification $I_{(+)}^{(3)} := J^{(+)}$. To this end, we discuss the situation in $(1, 1)$ superspace [8]. This discussion is general, only the later applications in the example section will be limited to $4D$. We replace spinor derivatives according to

$$\bar{\mathbb{D}}_{\pm} \rightarrow D_{\pm} + iQ_{\pm}, \quad \mathbb{D}_{\pm} \rightarrow D_{\pm} - iQ_{\pm}. \quad (3.13)$$

The general form of the $(2, 2)$ sigma model reduced to $(1, 1)$ has a Lagrangian that reads

$$\mathcal{L} = D_+X^A\mathbb{E}_{AB}(X)D_-X^C + \Psi_+^R K_{RL}\Psi_-^L := \mathcal{L}_1 + \mathcal{L}_2, \quad (3.14)$$

⁴This acronym stands for “bihermitean local product” and refers to a $2D$ sigma model with chiral and twisted chiral superfields only.

⁵Corresponding to $(+)$ -supersymmetries. The general case also involves $(-)$ -supersymmetries.

where $\mathbb{E}_{AB} := \mathbb{G}_{AB} + \mathbb{B}_{AB}$ and we have completed the square for the spinor auxiliary fields ψ_{\pm} and defined

$$\begin{aligned}\Psi_+^R &:= \psi_+^R - D_+ X^A J_{(+)A}^R \\ \Psi_-^L &:= \psi_-^L - J_{(-)A}^L D_- X^A .\end{aligned}\tag{3.15}$$

Assume that we have found the additional transformations of the $(1, 1)$ coordinates generated by the $SU(2)$ set of complex structures $I_{(+)}^{(\mathfrak{a})}$ as in (3.12). These transformations leave the \mathcal{L}_1 part of the action invariant. We would now like to extend them to symmetries of the full action and subsequently check if the full set can come from transformations of the $(2, 2)$ semichiral fields.

There are obvious symmetries we can write down, field equation symmetries (also called Zilch symmetries), but there are many possibilities.

3.1 $N = (2, 2)$

To get a guide to the correct form, we use the fact that we know that one of the symmetries has the correct properties; the one generated by $I_{(+)}^{(3)} = J_{(+)}$.

The full action is invariant under the following transformations:

$$\begin{aligned}\delta X^L &= \epsilon^+ J D_+ X^L \\ \delta X^R &= \epsilon^+ \psi_+^R \\ \delta \psi_-^L &= \epsilon^+ J D_+ \psi_-^L \\ \delta \psi_+^R &= \epsilon^+ D_+^2 X^R ,\end{aligned}\tag{3.16}$$

where J is the canonical complex structure $diag(i, -i)$. They give

$$\begin{aligned}\delta \mathcal{L} &= -2\delta X^A \nabla_+^{(-)} D_- X^B \mathbb{G}_{AB} + \delta \Psi_+^R K_{RL} \Psi_-^L + \Psi_+^R \delta K_{RL} \Psi_-^L + \Psi_+^R K_{RL} \delta \Psi_-^L \\ &= -2\epsilon^+ J_{(+)C}^A D_+ X^C \nabla_+^{(-)} D_- X^B \mathbb{G}_{AB} + \Psi_+^R \left(K_{RL} \delta \Psi_-^L + 2\epsilon^+ \nabla_+^{(-)} D_- X^B \mathbb{G}_{RB} \right) \\ &\quad + \delta \Psi_+^R K_{RL} \Psi_-^L + \Psi_+^R \delta K_{RL} \Psi_-^L ,\end{aligned}\tag{3.17}$$

where capital letters from the beginning of the alphabet takes on all values L and R . The first of the terms in the last line is the variation of \mathcal{L}_1 under the $J_{(+)}^{(3)}$ symmetry and vanishes in the action. We evaluate the remaining terms using

$$\begin{aligned}\delta \Psi_-^L &= \epsilon^+ J D_+ \Psi_-^L + \epsilon^+ J_{(-)R'}^L D_- \Psi_+^{R'} - \epsilon^+ J_{(-)AR'}^L \Psi_+^{R'} D_- X^A \\ &\quad + \epsilon^+ [J_{(+)}, J_{(-)}]_A^L \left(D_+ D_- X^A + D_+ X^B \Gamma_{BC}^{(-) A} D_- X^C \right) \\ \delta \Psi_+^R &= \epsilon^+ J_{(+)R'}^R D_+ \Psi_+^{R'} - \epsilon^+ J_{(+)AR'}^R \Psi_+^{R'} D_+ X^A\end{aligned}\tag{3.18}$$

which follows from (3.16). (Here $J_{(-)AR}^L$ denotes the derivative $(J_{(-)A})_{,R}$ etc.). Since

$$\mathbb{G}_{AB} = \Omega_{AC} [J_{(+)}, J_{(-)}]_B^C \quad (3.19)$$

the second line may be rewritten as

$$\epsilon^+ \sigma^{LC} \mathbb{G}_{CA} \nabla_+^{(-)} D_- X^A, \quad (3.20)$$

where we use the notation $\sigma^{LC} \Omega_{CD} = \delta_D^C$. This part of the variation will cancel the covariant derivative term multiplying Ψ_+^R in (3.17). We are left with

$$\delta \mathcal{L} = -\frac{1}{2} \epsilon^+ D_- \left(\Psi_+^R C_{RR'} \Psi_+^{R'} \right) + \epsilon^+ D_+ \left(\Psi_+^R K_{RL} J \Psi_-^L \right),$$

which ensures invariance of the action. Here where $C_{RR'}$ is the commutator with the canonical complex structure $C_{RR'} := [J, K_{RR'}]$.

In deriving (3.21) heavy use is made of integrability and covariant constancy of $J_{(\pm)}$ as well as their explicit expressions [1].

We note that the transformations that leave the action invariant may be written

$$\begin{aligned} \delta X^A &= \epsilon^+ J_{(+)B}^A D_+ X^B + \delta_R^A \epsilon^+ \Psi_+^R \\ \delta \Psi_-^L &= \epsilon^+ J D_+ \Psi_-^L + \epsilon^+ J_{(-)R'}^L D_- \Psi_+^{R'} - \epsilon^+ J_{(-)AR'}^L \Psi_+^{R'} D_- X^A \\ &\quad - 2\epsilon^+ K^{LR} \mathbb{G}_{RA} \nabla_+^{(-)} D_- X^A \\ \delta \Psi_+^R &= \epsilon^+ J_{(+)R'}^R D_+ \Psi_+^{R'} - \epsilon^+ J_{(+)AR'}^R \Psi_+^{R'} D_+ X^A \end{aligned} \quad (3.21)$$

3.2 $N=(4, 4)$

We now investigate if our additional super symmetries (4.56) can be written in terms of transformations of semichiral fields in $(2, 2)$ superspace. In our $(1, 1)$ language, the relation to semichirals is given by (3.15) in (X^L, X^R) coordinates⁶. We denote a generic complex structure by I_B^A and write the X transformations as

$$\delta X^A = \epsilon^+ [I_{(+)B}^A D_+ X^B + M_R^A \Psi_+^R],$$

where \mathcal{L}_1 is assumed to be invariant (up to total derivatives) under the first transformation on the RHS. Note that $M_L^A = 0$. The formula (3.22) is of the form in (3.21) and the most general expression compatible with dimensions and symmetries. From (3.15) we have that

⁶The reduction of $(\mathbb{X}^L, \mathbb{X}^R)$.

$$\begin{aligned}
\delta\Psi_+^{\dot{R}} &= \epsilon^+ \left[\left(I_{(+)\dot{R}}^{\dot{R}} - [M, J_{(+)}]_{\dot{R}}^{\dot{R}} \right) D_+ \Psi_+^R + \left(I_{+A,\dot{R}}^{\dot{R}} + \mathcal{M}(M, J_{(+)})_{\dot{R}A}^{\dot{R}} \right) D_+ X^A \Psi_+^R \right. \\
&\quad \left. - M_{\dot{R},\dot{R}'}^{\dot{R}} \Psi_+^{R'} \Psi_+^R \right] \\
\delta\Psi_-^{\dot{L}} &= \epsilon^+ \left[-[M, J_{(-)}]_{\dot{R}}^{\dot{L}} D_- \Psi_+^R + \mathcal{M}(M, J_{(-)})_{\dot{R}A}^{\dot{L}} D_- X^A \Psi_+^R \right. \\
&\quad + (I_{(+)\dot{L}}^{\dot{L}} - M_{\dot{R}}^{\dot{L}} J_{(+)\dot{L}}^R) D_+ \Psi_-^L + (I_{(+)\dot{A},\dot{L}}^{\dot{L}} - M_{\dot{R}}^{\dot{L}} J_{(+)\dot{A},\dot{L}}^R) D_+ X^A \Psi_-^L \\
&\quad \left. - M_{\dot{R},\dot{L}}^{\dot{L}} \Psi_-^L \Psi_+^R + \left([I_{(+), J_{(-)}]_{\dot{A}}^{\dot{L}} - M_{\dot{R}}^{\dot{L}} [J_{(+), J_{(-)}]_{\dot{A}}^R \right) \nabla_+^{(-)} D_- X^A \right]
\end{aligned} \tag{3.22}$$

where the Magri-Morosi concomitant for two endomorphisms I and J reads⁷

$$\mathcal{M}(I, J)_{BD}^A = I_B^F J_{D,F}^A - J_D^F I_{B,F}^A - I_F^A J_{D,B}^F + J_F^A I_{B,D}^F . \tag{3.23}$$

When $M_{\dot{R}}^A = \delta_{\dot{R}}^A$ and $I_+ = J_{(+)}$ these transformations reduce to (3.21).

From invariance of (3.14), we find a number of relations. First, raising and lowering indices on M with K_{RL} ,

$$\begin{aligned}
M_{L[R,\dot{R}]} - M_{[R\dot{R}],L} &= 0 \\
M_{[R\dot{R}]} &= -\frac{1}{2} K_{\dot{R}\dot{L}} [I_{(+), J_{(-)}]_{\dot{A}}^{\dot{L}} \mathbb{G}^{AL} K_{LR} \\
M_{\dot{R}}^R &= -\frac{1}{2} K_{\dot{R}\dot{L}} [I_{(+), J_{(-)}]_{\dot{A}}^{\dot{L}} \mathbb{B}^{AR}
\end{aligned} \tag{3.24}$$

Note that only the antisymmetric part of $M_{R\dot{R}}$ is determined by this⁸. The D_- terms in the variation of \mathcal{L} are

$$\Psi_+^{\dot{R}} K_{\dot{R}\dot{L}} \epsilon^+ \left(-[M, J_{(-)}]_{\dot{R}}^{\dot{L}} D_- \Psi_+^R + \mathcal{M}(M, J_{(-)})_{\dot{R}A}^{\dot{L}} D_- X^A \Psi_+^R \right)$$

For this to yield a total D_- derivative, we shall need

$$-K_{(\dot{R}|L|} [M, J_{(-)}]_{\dot{R}}^{\dot{L}} = [J, M]_{(\dot{R}\dot{R})} + C_{(\dot{R}|\dot{R}|} M_{\dot{R}}^R = 0 \tag{3.25}$$

and

$$K_{[\dot{R}|L|} \mathcal{M}(M, J_{(-)})_{\dot{R}A}^{\dot{L}} D_- X^A = -\frac{1}{2} D_- (K_{[\dot{R}|L|} [M, J_{(-)}]_{\dot{R}}^{\dot{L}}) \tag{3.26}$$

⁷Originally defined for a Poisson structure P and a Nijenhuis tensor N when it reads [10]

$$C_m^{kj} = P^{lj} N_{m,l}^k + P^{kl} N_{m,l}^k - N_m^l P_{,l}^{kj} + N_l^j P_{,m}^{kl} - P^{lj} N_{l,m}^k$$

and is only a tensor when $[P, N] = 0$.

⁸The RHS of the equation containing $M_{[R\dot{R}]}$ is antisymmetric due to hermiticity conditions.

Similarly, for the D_+ terms to yield a total derivative, we need

$$K_{LR}I_{\dot{R}}^R - K_{\dot{R}L}I_L^L + K_{LR}[\tilde{J}, M]_{\dot{R}}^R + C_{LL}K^{LR}M_{[R, \dot{R}]} = 0$$

and

$$\begin{aligned} & \left(I_{A, \dot{R}}^R + \mathcal{M}(M, J_{(+)}^R)_{\dot{R}A}^R \right) K_{RL} + K_{\dot{R}LB}I_A^B + K_{\dot{R}L} \left(I_{A, L}^L - M_{\dot{R}J_{(+)}A, L}^L \right) \\ &= \left((I_L^L - M_{\dot{R}J_{(+)}L}^L) K_{\dot{L}\dot{R}} \right)_{,A} \end{aligned} \quad (3.27)$$

where we have used (3.24) and the explicit form of $J_{(+)}$.

4 The $S^3 \times S^1$ model

4.1 Duality

In this section we briefly recapitulate the dualisation of the BiLP formulation of the $SU(2) \times U(1)$ WZW model [5], albeit in a different version.

We start from the following BiLP potential which gives a sigma model with target space geometry $S^3 \times S^1$;

$$K = -\ln \hat{\chi} \ln \hat{\bar{\chi}} + \int^{\frac{\hat{\phi}\hat{\bar{\phi}}}{\hat{\chi}\hat{\bar{\chi}}}} dq \frac{\ln(1+q)}{q}, \quad (4.28)$$

where $\hat{\phi}$ is chiral, $\bar{\mathbb{D}}_{\pm}\hat{\phi} = 0$, and $\hat{\chi}$ is twisted chiral, $\bar{\mathbb{D}}_{+}\hat{\chi} = 0 = \mathbb{D}_{-}\hat{\chi}$. The potential satisfies the Laplacian

$$K_{\hat{\phi}\hat{\bar{\phi}}} + K_{\hat{\chi}\hat{\bar{\chi}}} = 0, \quad (4.29)$$

and hence the model has (4,4) supersymmetry [8]. Changing coordinates to new chiral and twisted chiral fields, $\phi = \ln \hat{\phi}$, $\chi = \ln \hat{\chi}$, results in

$$K \rightarrow K = -\chi \bar{\chi} + \int^{\phi + \bar{\phi} - \chi - \bar{\chi}} dq \ln(1 + e^q), \quad (4.30)$$

and makes it amenable to dualisation of the translation symmetry⁹

$$\phi \rightarrow \phi + \lambda, \quad \chi \rightarrow \chi + \lambda. \quad (4.31)$$

⁹This is equivalent to dualising the scaling symmetry of (4.28). These isometries both commute with the extra supersymmetry [11], [12].

To apply the gauging prescription of [13] we add a term $\alpha(\chi - \bar{\chi})(\phi - \bar{\phi})$, which represents a constant B -term, to the Lagrangian and rewrite the potential, up to generalized Kähler gauge transformations, as

$$\frac{1}{2}(\chi - \bar{\chi})^2 + \alpha(\chi - \bar{\chi})(\phi - \bar{\phi}) + \int^{\phi + \bar{\phi} - \chi - \bar{\chi}} dq \ln(1 + e^q) . \quad (4.32)$$

Following [13], we find the first order action (in a WZ gauge);

$$-\frac{1}{2}V_\chi^2 - \alpha V_\chi V_\phi - V' X' - V_\phi X_\phi - V_\chi X_\chi + \int^{V'} dq \ln(1 + e^q) , \quad (4.33)$$

where V_ϕ, V_χ and V' are the Large Vector Multiplet (LVM) fields [13]¹⁰, and the Lagrange multipliers are combinations of semichiral fields

$$\begin{aligned} X_\phi &= \frac{i}{2}(\ell - \bar{\ell} - r + \bar{r}) \\ X_\chi &= \frac{i}{2}(-\ell + \bar{\ell} - r + \bar{r}) \\ X' &= \frac{1}{2}(\ell + \bar{\ell} - r - \bar{r}) . \end{aligned} \quad (4.34)$$

Eliminating the LVM and massaging the integral we find the dual semichiral action in the form

$$-\frac{1}{2\alpha^2}X_\phi^2 + \frac{1}{\alpha}X_\phi X_\chi - \int^{X'} dq \ln(e^q - 1) . \quad (4.35)$$

This is the potential that is expected to have additional supersymmetries due to those of the dual BiLP model.

4.2 The geometry

The reduction of a semichiral model to $(1, 1)$ superspace may be expressed in several useful coordinate systems. E.g., the (X^L, X^R) coordinates directly obtained in the reduction is related to the (X, Y) system where $J_{(+)}$ is canonical via a coordinate transformation [14],[1].

¹⁰In pure gauge they become

$$\begin{aligned} V_\phi &= i(\bar{\phi} - \phi) \\ V_\chi &= i(\bar{\chi} - \chi) \\ V' &= \phi + \bar{\phi} - \chi - \bar{\chi} . \end{aligned}$$

We now derive the metric in (X, Y) coordinates for (4.35). To this end we first calculate the various ingredient matrices according to the formulae in [1].

Without loss of generality, we set α to -1 in (4.35) and drop the prime on X' . The Y coordinates are defined to be $K_L =: Y$. We find

$$\begin{aligned}
-y &= K_\ell = \frac{i}{2}X_\chi + \frac{1}{2}\ln(e^X - 1) = \frac{1}{4}(\ell - \bar{\ell}) + \frac{1}{4}(r - \bar{r}) + \frac{1}{2}\ln(e^X - 1) \\
&\Rightarrow \\
X &= \ln(1 + e^{-(y+\bar{y})}) \\
r + \bar{r} &= \ell + \bar{\ell} - 2\ln(1 + e^{-(y+\bar{y})}) , \\
r - \bar{r} &= -2(y - \bar{y}) - (\ell - \bar{\ell})
\end{aligned} \tag{4.36}$$

The relevant matrices of derivatives of K are

$$\begin{aligned}
K_{LL} &= -\frac{1}{4N} \begin{pmatrix} E & 1 \\ 1 & E \end{pmatrix} , & -K_{RR} &= \frac{1}{4N} \begin{pmatrix} 2-E & M \\ 4N+1 & 2-E \end{pmatrix} \\
-K_{LR} &= -\frac{1}{4N} \begin{pmatrix} 1 & E \\ E & 1 \end{pmatrix} , & -K^{RL} &= \frac{1}{e^X} \begin{pmatrix} 1 & -E \\ -E & 1 \end{pmatrix} \\
-C_{LL} &= \frac{2i}{4N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , & -C_{RR} &= \frac{2iM}{4N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{aligned} \tag{4.37}$$

where we have introduced the notation

$$N := e^{-(y+\bar{y})} = e^X - 1 , \quad E := 2e^X - 1 , \quad M := 4N + 1 , \tag{4.38}$$

for combinations that will occur frequently in our formulae. The metric and B -field in (X, Y) coordinates can be calculated from the formulae in [1]:

$$\begin{aligned}
-\mathbb{E}_{LL} &= J(K_{LL}K^{LR}JK_{RL} - K_{LR}JK^{RL}K_{LL} - K_{LL}K^{LR}C_{RR}K^{RL}K_{LL}) = -2\sigma_1 \\
\mathbb{E}_{LY} &= J(K_{LR}JK^{RL} + K_{LL}K^{LR}C_{RR}K^{RL}) = -\frac{1}{e^X} \begin{pmatrix} 2-e^X & -2E \\ -2E & 2-e^X \end{pmatrix} \\
\mathbb{E}_{YL} &= J(-K^{LR}JK_{RL} + K^{LR}C_{RR}K^{RL}K_{LL}) = \frac{1}{e^X} \begin{pmatrix} 2-e^X & 2E \\ 2E & 2-e^X \end{pmatrix} \\
-\mathbb{E}^{YY} &= -JK^{LR}C_{RR}K^{RL} = -\frac{2M}{e^X}\sigma_1 .
\end{aligned}$$

Note that the tensor \mathbb{E} depends on $y + \bar{y}$ only. From the formulae for $\mathbb{E} = \mathbb{G} + \mathbb{B}$, it follows

that the metric is¹¹

$$\mathbb{G} = \frac{2}{e^X} \begin{pmatrix} e^{-X} & E \\ E & M \end{pmatrix} \otimes \sigma_1 , \quad (4.39)$$

with inverse

$$\mathbb{G}^{-1} = -\frac{e^X}{2N} \begin{pmatrix} -M & E \\ E & -e^X \end{pmatrix} \otimes \sigma_1 =: -\frac{e^X}{2N} \mathbf{h} \otimes \sigma_1 .$$

Also

$$\mathbb{B}_{YL} = (2e^{-X} - 1)\mathbb{1} . \quad (4.40)$$

In these coordinates $J_{(+)}$ is canonical while

$$\begin{aligned} J_{(-)} &= e^{-X} \begin{pmatrix} (2 - e^X)J - 2E\sigma_2 & -2M\sigma_2 \\ 2e^X\sigma_2 & (2 - e^X)J + 2E\sigma_2 \end{pmatrix} \\ &=: e^{-X} (2\mathbf{j} \otimes \sigma_2 + (2 - e^X) \otimes i\sigma_3) . \end{aligned} \quad (4.41)$$

where we use that $J = i\sigma_3$.

4.3 Additional SUSY in (1, 1)

4.3.1 Deriving the transformations on the (1, 1) coordinate fields

In the original BiLP, (4.28) the additional super symmetries read [8]

$$\begin{aligned} \delta\hat{\phi} &= \bar{\epsilon}^+ \bar{\mathbb{D}}_+ \hat{\chi} + \bar{\epsilon}^- \bar{\mathbb{D}}_- \hat{\chi} \\ \delta\hat{\bar{\phi}} &= \epsilon^+ \mathbb{D}_+ \hat{\chi} + \epsilon^- \mathbb{D}_- \hat{\chi} \\ \delta\hat{\chi} &= -\bar{\epsilon}^+ \bar{\mathbb{D}}_+ \hat{\phi} - \epsilon^- \mathbb{D}_- \hat{\phi} \\ \delta\hat{\bar{\chi}} &= -\epsilon^+ \mathbb{D}_+ \hat{\phi} - \bar{\epsilon}^- \bar{\mathbb{D}}_- \hat{\phi} , \end{aligned} \quad (4.42)$$

and in the transformed version (4.30) they become

$$\begin{aligned} \delta\phi &= e^{\bar{\chi}-\phi} \bar{\epsilon}^+ \bar{\mathbb{D}}_+ \bar{\chi} + e^{\chi-\phi} \bar{\epsilon}^- \bar{\mathbb{D}}_- \chi \\ \delta\bar{\phi} &= e^{\chi-\bar{\phi}} \epsilon^+ \mathbb{D}_+ \chi + e^{\bar{\chi}-\bar{\phi}} \epsilon^- \mathbb{D}_- \bar{\chi} \\ \delta\chi &= -e^{\bar{\phi}-\chi} \bar{\epsilon}^+ \bar{\mathbb{D}}_+ \bar{\phi} - e^{\phi-\chi} \epsilon^- \mathbb{D}_- \phi \\ \delta\bar{\chi} &= -e^{\phi-\bar{\chi}} \epsilon^+ \mathbb{D}_+ \phi - e^{\bar{\phi}-\bar{\chi}} \bar{\epsilon}^- \bar{\mathbb{D}}_- \bar{\phi} . \end{aligned} \quad (4.43)$$

¹¹Since a lot of the objects have $2D$ complex submatrices, it is convenient to introduce the Pauli matrices σ_i and write matrices as direct products.

These relations survive in the $(1, 1)$ reduction with $\mathbb{D}_\pm \rightarrow D_\pm$. From (4.43) we then read off the additional complex structures according to

$$\delta\varphi = \frac{1}{2} \left[\left(J_{(\pm)}^{(1)} + iJ_{(\pm)}^{(2)} \right) \epsilon^\pm D_\pm \varphi + \left(J_{(\pm)}^{(1)} - iJ_{(\pm)}^{(2)} \right) \bar{\epsilon}^\pm D_\pm \varphi \right] \quad (4.44)$$

For the $J_{(+)}^{(\mathbf{a})}$ we find

$$J_{(+)}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & e^{\bar{\chi}-\phi} \\ 0 & 0 & e^{\chi-\bar{\phi}} & 0 \\ 0 & -e^{\bar{\phi}-\chi} & 0 & 0 \\ -e^{\phi-\bar{\chi}} & 0 & 0 & 0 \end{pmatrix} \quad (4.45)$$

$$J_{(+)}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & ie^{\bar{\chi}-\phi} \\ 0 & 0 & -ie^{\chi-\bar{\phi}} & 0 \\ 0 & -ie^{\bar{\phi}-\chi} & 0 & 0 \\ ie^{\phi-\bar{\chi}} & 0 & 0 & 0 \end{pmatrix}, \quad (4.46)$$

with $J_{(+)}^{(3)} = J$, the canonical complex structure.

We would like to see what these complex structures look like in $(1, 1)$ coordinates related to the semichiral description. While T-dual formulations are not in general related by coordinate transformations, they are in this case due to the special choice of the isometry direction; we have dualised along the common $U(1)$. We now need to find the coordinate transformation. To this end, we note that the relations (A.74), derived in the appendix,

$$\begin{aligned} V_\phi &= X_\chi + X_\phi \\ V_\chi &= X_\phi \\ V &= \ln(e^X - 1). \end{aligned} \quad (4.47)$$

do not completely determine the transformations. Identifying the LHS with BiLP fields (WZ-gauge) and writing out the RHS we have

$$\begin{aligned} i(\bar{\phi} - \phi) &= -i(r - \bar{r}) \\ i(\bar{\chi} - \chi) &= \frac{i}{2}(\ell - \bar{\ell} - r + \bar{r}) \\ \phi + \bar{\phi} - \chi - \bar{\chi} &= \ln(e^{\frac{1}{2}(\ell + \bar{\ell} - r - \bar{r})} - 1). \end{aligned} \quad (4.48)$$

It turns out to be most convenient to identify the coordinate transformation to (X, Y) coordinates where the complex structure derived from the semi side, $J_{(+)}$, is canonical¹².

¹²The map of $J_{(\pm)}$ under duality is discussed in [15] where the dual model appears in some preferred coordinates. We have not investigated the relation to the present coordinates.

A coordinate transformation to (X^L, X^R) coordinates will then give a non-canonical $J_{(+)}$. Comparing this to (4.36), we identify

$$\begin{aligned} y &= \chi - \phi \\ \ell - \bar{\ell} &= 2(\bar{\chi} - \chi) - (\bar{\phi} - \phi) , \end{aligned} \quad (4.49)$$

but $\ell + \bar{\ell}$ is left undetermined. In both coordinate systems we have $J_{(+)}^{(3)} = J$. Requiring that the coordinate transformation takes the canonical complex structure into itself determines $\ell + \bar{\ell} = \phi + \bar{\phi} - 2(\bar{\chi} - \chi)$, which results in

$$\ell = \phi - 2\chi . \quad (4.50)$$

This gives the transformation Jacobian

$$\mathbb{A} = \begin{pmatrix} \frac{\partial L}{\partial \phi} & \frac{\partial L}{\partial \chi} \\ \frac{\partial Y}{\partial \phi} & \frac{\partial Y}{\partial \chi} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & -2\mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} , \quad (4.51)$$

with inverse

$$\mathbb{A}^{-1} = - \begin{pmatrix} \mathbb{1} & 2\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} , \quad (4.52)$$

These transformations correctly relates the BiLP metric derived from (4.30) to the semi metric (4.39). We now write the extra complex structures $J_{(+)}^{(\mathfrak{a})}$ as

$$J_{(+)}^{(\mathfrak{a})} = \begin{pmatrix} 0 & \mathbb{A}^{(\mathfrak{a})} \\ -(\mathbb{A}^{(\mathfrak{a})})^{-1} & 0 \end{pmatrix} \quad (4.53)$$

for $\mathfrak{a} = 1, 2$, with

$$\begin{aligned} \mathbb{A}^{(1)} &= \begin{pmatrix} 0 & e^{\bar{\chi} - \phi} \\ e^{\chi - \bar{\phi}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{\varphi + y} \\ e^{-\varphi + \bar{y}} & 0 \end{pmatrix} \\ \mathbb{A}^{(2)} &= \begin{pmatrix} 0 & ie^{\bar{\chi} - \phi} \\ -ie^{\chi - \bar{\phi}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & ie^{\varphi + y} \\ -ie^{-\varphi + \bar{y}} & 0 \end{pmatrix} , \end{aligned} \quad (4.54)$$

where

$$\varphi := (\ell - \bar{\ell}) + (y - \bar{y}) . \quad (4.55)$$

The expressions for $J_{(+)}^{(\mathfrak{a})}$ in (X, Y) coordinates then become

$$\begin{aligned} J_{(+)}^{(\mathfrak{a})} &= \begin{pmatrix} -E & -M \\ e^X & E \end{pmatrix} \otimes \mathbb{A}^{(\mathfrak{a})} =: \mathbf{j} \otimes \mathbb{A}^{(\mathfrak{a})} , \\ \mathbf{j}^2 &= -N , \quad (\mathbb{A}^{(\mathfrak{a})})^2 = N^{-1} , \end{aligned} \quad (4.56)$$

where we again use the notation in (4.38). The complex structures $J_{(+)}^{(\mathfrak{a})}$ preserve the metric (4.39), as confirmed by an explicit calculation.

4.3.2 (X^L, X^R) coordinates

Using (4.37) in the Jacobian

$$\begin{aligned}\mathbb{A} &= \begin{pmatrix} \mathbb{1} & 0 \\ K_{LL} & K_{LR} \end{pmatrix} = \frac{1}{4N} \begin{pmatrix} 4N & 0 \\ -(E\mathbb{1} + \sigma_1) & \mathbb{1} + E\sigma_1 \end{pmatrix}, \\ \mathbb{A}^{-1} &= \begin{pmatrix} \mathbb{1} & 0 \\ -K^{RL}K_{LL} & K^{RL} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ \sigma_1 & -e^{-X}(\mathbb{1} - E\sigma_1) \end{pmatrix},\end{aligned}\quad (4.57)$$

we find the expressions for $J_{(+)}^{(\mathfrak{A})}$ in left right coordinates (X^L, X^R) :

$$\begin{aligned}J_{(+)}^{(\mathfrak{A})} &= \\ &\frac{1}{4N} \left\{ \begin{pmatrix} E & -M \\ e^{-X} & -e^{-X}E \end{pmatrix} \otimes \mathbb{A}^{(A)} + \begin{pmatrix} M & -ME \\ e^{-X}E & -e^{-X}E^2 \end{pmatrix} \otimes \mathbb{A}^{(A)}\sigma_1 \right. \\ &\quad \left. + \frac{N}{e^X} \left[\begin{pmatrix} 0 & 0 \\ 1 & -E \end{pmatrix} \otimes \bar{\mathbb{A}}^{(A)} + \begin{pmatrix} 0 & 0 \\ E & -1 \end{pmatrix} \otimes \bar{\mathbb{A}}^{(A)}\sigma_1 \right] \right\},\end{aligned}\quad (4.58)$$

where $Y(X^L, X^R)$ is given by the relations in (4.36).

5 Examples

5.1 Hyperkähler

We want to show that there are hyperkähler solutions to our problem in $4D$. To this end we note that, when c is constant we have available the following hyperkähler structure [16];

$$\mathcal{I} := J_{(+)} , \quad \mathcal{J} := \frac{1}{\sqrt{1-c^2}}(J_{(-)} + cJ_{(+)}), \quad \mathcal{K} := \frac{1}{2\sqrt{1-c^2}}[J_{(+)}, J_{(-)}]. \quad (5.59)$$

The relations (3.24) determine M in the three cases according to

$$\begin{aligned}\mathcal{I}: \quad & M_{\dot{R}}^R = \delta_{\dot{R}}^R, \quad M_{[\dot{R}R]} = 0 \\ \mathcal{J}: \quad & M_{\dot{R}}^R = \frac{c \delta_{\dot{R}}^R}{\sqrt{1-c^2}}, \\ \mathcal{K}: \quad & M_{\dot{R}}^R K_{LR} = -\frac{1}{\sqrt{1-c^2}} K_{\dot{R}L} J_{(-)L}^L = -\frac{1}{\sqrt{1-c^2}} J K_{\dot{R}L} \\ & M_{[\dot{R}R]} = -\frac{1}{\sqrt{1-c^2}} K_{\dot{R}L} J_{(-)R}^L = -\frac{1}{\sqrt{1-c^2}} C_{\dot{R}R}\end{aligned}\quad (5.60)$$

Each case satisfies the first relation in (3.24) (provided that c is constant) .

The conditions (3.25) is satisfied by the hyperkähler structure (5.60). The relation (3.26) is satisfied for \mathcal{I} and \mathcal{J} by direct insertion. For \mathcal{K} we determine the full

$$M_{\dot{R}}^L = -\frac{1}{\sqrt{1-c^2}} K^{LR} J K_{R\dot{R}} \quad (5.61)$$

and find that

$$K_{[\dot{R}|L]}[M, J_{(-)}]_{R]}^L = 0 , \quad (5.62)$$

and the issue becomes the vanishing of \mathcal{M} . This is again confirmed by direct insertion of the \mathcal{K} expressions from (5.60).

As a final check we also find that the relations (3.27) and (3.27) are indeed satisfied for \mathcal{I}, \mathcal{J} and \mathcal{K} .

5.2 $SU(2) \otimes U(1)$

Using (4.56) and (4.41) we find that in (X, Y) coordinates

$$[J_+^{(\mathfrak{a})}, J_-] = -2e^{-X} \left((2 - e^X) \mathbf{j} \otimes (b^{(\mathfrak{a})} \sigma_1 + a^{(\mathfrak{a})} \sigma_2) + 2N \mathbb{1} \otimes a^{(\mathfrak{a})} i \sigma_3 \right) , \quad (5.63)$$

where \mathbf{j} is defined in (4.41). Using (4.54) we have defined

$$\begin{aligned} \mathbb{A}^{(1)} &=: a^{(1)} \sigma_1 - b^{(1)} \sigma_2 =: \frac{1}{\sqrt{N}} (\cos \psi \sigma_1 - \sin \psi \sigma_2) \\ \mathbb{A}^{(2)} &=: a^{(2)} \sigma_1 - b^{(2)} \sigma_2 =: \frac{1}{\sqrt{N}} (-\sin \psi \sigma_1 - \cos \psi \sigma_2) , \end{aligned} \quad (5.64)$$

and

$$i\psi := (\ell - \bar{\ell}) + \frac{3}{2}(y - \bar{y}) . \quad (5.65)$$

As is clear from (3.24), we shall need $[J_+^{(\mathfrak{a})}, J_-] \mathbb{G}^{-1}$. We find

$$[J_+^{(\mathfrak{a})}, J_-] \mathbb{G}^{-1} = 2a^{(A)} \mathbf{h} \otimes \sigma_2 + (2 - e^X) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (b^{(\mathfrak{a})} \mathbb{1} - a^{(\mathfrak{a})} i \sigma_3) , \quad (5.66)$$

where \mathbf{h} is defined in (4.40). In (X^L, X^R) coordinates this becomes

$$(2e^{-X} - 1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes b^{(\mathfrak{a})} (\mathbb{1} - E \sigma_1) + \begin{pmatrix} 2M & E \\ E & 0 \end{pmatrix} \otimes a^{(\mathfrak{a})} \sigma_2 - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes a^{(\mathfrak{a})} i \sigma_3 \quad (5.67)$$

We read off the matrices relevant to (3.24)

$$\begin{aligned}
M_{[R\dot{R}]} &= \frac{1}{2} K_{\dot{R}\dot{L}} \left([J_+^{(\mathfrak{a})}, J_-] \mathbb{G}^{-1} \right)^{\dot{L}L} K_{LR} = -\frac{Me^X a^{(A)}}{4N} \sigma_2 \\
M_{LR} &= \frac{1}{2} K_{R\dot{L}} \left([J_+^{(\mathfrak{a})}, J_-] \mathbb{G}^{-1} \right)^{\dot{L}R} K_{RL} \\
&= -\frac{1}{2} \frac{e^X}{4N} \left[(2e^{-X} - 1) b^{(\mathfrak{a})} (1 + E\sigma_1) + a^{(\mathfrak{a})} (E\sigma_2 - i\sigma_3) \right]
\end{aligned} \tag{5.68}$$

We find that the quantities in (5.68) indeed satisfy the first relation in (3.24). Proceeding to (3.25) and (3.26), we find that (3.25) is also satisfied using (5.68), and that the $b^{(\mathfrak{a})}$ terms in (3.26) cancel. However, the remaining terms in (3.26) must satisfy

$$\begin{aligned}
(M_{[r}^F K_{\bar{r}]F})_R &= 0 \\
M_{r[r}^r K_{\bar{r}]L} &= 0,
\end{aligned} \tag{5.69}$$

where knowledge of the form of $J_{(-)}$ along with partial information from (5.68) has been used. While the first of these equations determines the remaining parts of M_R^L , the second equations must be identically satisfied by M_R^R in (5.68). This is not the case.

6 Discussion

We have extended the $(1, 1)$ formulation of semichiral sigma models to allow for a treatment of extra super symmetries with on-shell closure. To exemplify the general method we have shown that a set of hyperkähler geometries arise as solutions of the conditions for extra supersymmetry. We have further constructed the extra super symmetries in a semichiral models dual to a BiLP model with “manifest” $(4, 4)$ susy on the BiLP side. This model fails the criteria for the additional supersymmetry to be manifest as transformations of $(2, 2)$ semichirals. Another way of saying this is that the $(4, 4)$ supersymmetry is incompatible with the introduction of the $(2, 2)$ auxiliary spinor fields. The key ingredient in the analysis is to show that invariance of the action fails (on-shell closure of the algebra is ensured by construction). Note that the analysis shows that not even an extra supersymmetry of one handedness only is possible.

Our analysis is carried out at the $(1, 1)$ level, where conditions for additional supersymmetries are well established since thirty years [8].

An analysis at the $(2, 2)$ level already indicated that the remedy suggested in [3] will not work; a formulation including central charge transformations will typically display the original obstructions when we go on-shell.

A further indication of problems with an extra supersymmetry comes from dualisation procedure itself. One would expect the parent action, where the chirality constraints on the chiral and twisted chiral superfields have been relaxed, to have the extra supersymmetry. This would mean the the LVM gauge multiplet could carry extra supersymmetry. This was concluded to be impossible under fairly general assumptions in [2].

In view of this result, it is reasonable to conjecture that manifest extra supersymmetries involving semichiral fields together with a $4D$ target space is only possible in models including auxiliary fields such as in the $(4, 4)$ superspace setting of [17].

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A Duality in $(1, 1)$

In this section we reduce the action (4.33) (with $\alpha = -1$) to $(1, 1)$ and eliminate the LVM there instead. This makes clear the issue of coordinate transformations at the $(1, 1)$ level. We replace covariant derivatives according to

$$\bar{D}_\pm \rightarrow D_\pm + iQ_\pm, \quad D_\pm \rightarrow D_\pm - iQ_\pm. \quad (\text{A.70})$$

To facilitate the calculation we introduce the following notation:

$$\begin{aligned} Y_\pm^A &:= Q_\pm X_A, \\ Z^A &:= Q_+ Q_\pm X_A, \quad A = \phi, \chi, X \\ s^\ell &:= \ell + \bar{\ell}, \quad d^\ell := \ell - \bar{\ell} \\ s^r &:= r + \bar{r}, \quad d^r := r - \bar{r} \\ \Sigma &:= \psi + \bar{\psi} \\ \Lambda &:= \psi - \bar{\psi} \end{aligned} \quad (\text{A.71})$$

and define the $(1, 1)$ components of the LVM (in WZ gauge) as

$$\begin{aligned} V_\chi| &=: V_\chi, \quad Q_\pm V_\chi| =: (A + B)_\pm, \quad Q_+ Q_\pm V_\chi| =: F \\ V_\phi| &=: V_\phi, \quad Q_\pm V_\phi| =: B_\pm, \quad Q_+ Q_\pm V_\phi| =: G \\ V_X| &=: V_X, \quad Q_\pm V_X| =: C_\pm, \quad Q_+ Q_\pm V_X| =: H. \end{aligned} \quad (\text{A.72})$$

The Lagrangian becomes

$$\begin{aligned}
& F(V_\phi - V_\chi - X_\chi) + G(V_\chi - X_\phi) + H(\ln(1 + e^V) - X) \\
& - V_\phi Z^\phi - V_\chi Z^\chi - V' Z^X - (A_+ + Y_+^\chi)(A_- + Y_-^\chi) + Y_+^\chi Y_-^\chi \\
& + (B_+ + Y_+^\phi + Y_+^\chi)(B_- + Y_-^\phi + Y_-^\chi) - (Y_+^\phi + Y_+^\chi)(Y_-^\phi + Y_-^\chi) \\
& + \left(C_+ - Y_+^\chi \left(\frac{1 + e^V}{e^V} \right) \right) \left(\frac{e^V}{1 + e^V} \right) \left(C_- - \left(\frac{1 + e^V}{e^V} \right) Y_-^\chi \right) \\
& - Y_+^\chi Y_-^\chi \left(\frac{1 + e^V}{e^V} \right) .
\end{aligned} \tag{A.73}$$

Integrating out F, G, H gives the coordinate transformation

$$\begin{aligned}
V_\phi &= X_\chi + X_\phi \\
V_\chi &= X_\phi \\
V &= \ln(e^X - 1) .
\end{aligned} \tag{A.74}$$

Integrating V_ϕ, V_χ, V determines F, G, H in terms of the components of the semis:

$$\begin{aligned}
F &= Z^\phi \\
G &= Z^\phi + Z^\chi \\
H &= \left(\frac{1 + e^V}{e^V} \right) \left[\left(1 - \frac{1 + e^V}{e^V} \right) Y_+^\chi Y_-^\chi + Z^X \right] .
\end{aligned} \tag{A.75}$$

(No contribution from C_\pm terms et.c.. 1.5 formalism). Finally, integrating A, B, C again determines these fields in terms of the semi components. This leaves us with a purely semi Lagrangian;

$$\begin{aligned}
& Y_+^\chi Y_-^\chi - (Y_+^\phi + Y_+^\chi)(Y_-^\phi + Y_-^\chi) - Y_+^\chi Y_-^\chi \left(\frac{e^X}{e^X - 1} \right) \\
& - (X_\chi + X_\phi) Z^\phi - X_\phi Z^\chi - \ln(e^X - 1) Z^X
\end{aligned} \tag{A.76}$$

As a check that this agrees with the reduced semi action, we integrate out the auxiliary spinors Ψ_\pm and reconstruct the complex structures $J^{(\pm)}$. We shall need

$$\begin{aligned}
Y_+^\phi &= \frac{i}{2} [iD_+ s^\ell - \Lambda_+^r] \\
Y_-^\phi &= \frac{i}{2} [\Lambda_-^\ell - iD_- s^r] \\
Y_+^\chi &= -\frac{i}{2} [iD_+ s^\ell + \Lambda_+^r] \\
Y_-^\chi &= -\frac{i}{2} [\Lambda_-^\ell + iD_- s^r]
\end{aligned}$$

$$\begin{aligned}
Y_+^X &= \frac{1}{2} [iD_+ d^\ell - \Sigma_+^r] \\
Y_-^X &= \frac{1}{2} [\Sigma_-^\ell - iD_- d^r] \\
Z^\phi &= -\frac{1}{2} [D_+ \Sigma_-^\ell + D_- \Sigma_+^r] \\
Z^X &= \frac{1}{2} [D_+ \Sigma_-^\ell - D_- \Sigma_+^r] \\
Z^X &= \frac{i}{2} [D_+ \Lambda_-^\ell + D_- \Lambda_+^r]
\end{aligned} \tag{A.77}$$

From the variations we find:

$$\begin{aligned}
\delta \Lambda_+^r : \quad \Lambda_-^\ell &= 3iD_- s^r - 2i \left(\frac{e^X}{e^X - 1} \right) D_- X \\
\delta \Sigma_+^r : \quad \Sigma_-^\ell &= iD_- d^r - 2 \left(\frac{e^X - 1}{e^X} \right) D_- (X_\chi + 2X_\phi) \\
\delta \Lambda_-^\ell : \quad \Lambda_+^r &= -iD_+ s^\ell + 2i \left(\frac{e^X}{e^X - 1} \right) D_+ X \\
\delta \Sigma_-^\ell : \quad \Sigma_+^r &= iD_+ d^\ell + 2 \left(\frac{e^X - 1}{e^X} \right) D_+ X_\chi .
\end{aligned}$$

This implies that (in the notation of (4.38))

$$\begin{aligned}
\Psi_-^\ell &= i \frac{1}{4e^X N} [(1 + E^2)D_- \ell + 2ED_- \bar{\ell} - 2(4N + 1)ED_- r - 2(4N + 1)D_- \bar{r}] \\
\Psi_+^r &= i \frac{1}{4e^X N} [2ED_+ \ell + 2D_+ \bar{\ell} - (1 + E^2)D_+ r - 2ED_+ \bar{r}]
\end{aligned} \tag{A.78}$$

These are the correct expressions for these auxiliary fermions, as may be checked using the matrices (4.37) in the formulae for $J^{(\pm)}$ in [1].

B An alternative dual form

We have been studying the action (4.35). It can be cast into a different form which connects to the results in [18]. We perform a Legendre transformation of the right semichiral superfields (for $\alpha = 1$)

$$\tilde{K} = K(L, x, \bar{x}) - xr - \bar{x}\bar{r} \tag{B.79}$$

which together with the change $L \rightarrow -L$ (and some manipulations of the integral) brings the potential to the form found in [18]:

$$-(\ell - \bar{r})(\bar{\ell} - r) + \int^{r+\bar{r}} dq \ln(1 + e^q) . \tag{B.80}$$

We already know the metric in these (X^L, X^R) coordinates from [18]

$$\mathbb{G} = \begin{pmatrix} \sigma_1 & -Z \\ -Z & Z\sigma_1 \end{pmatrix}, \quad (\text{B.81})$$

where

$$Z := \frac{1}{1 + e^{r+\bar{r}}} = \frac{1}{1 + e^{\ell+\bar{\ell}-y-\bar{y}}}, \quad (\text{B.82})$$

and transformation to new (X, Y) coordinates reads

$$\begin{aligned} y &= K_\ell = \bar{\ell} - r \\ r &= \bar{\ell} - y. \end{aligned} \quad (\text{B.83})$$

The corresponding Jacobian is [1]

$$J = \begin{pmatrix} \mathbb{1} & 0 \\ -K^{RL}K_{LL} & K^{RL} \end{pmatrix}. \quad (\text{B.84})$$

We have

$$\begin{aligned} K_{LL} &= -\sigma_1, & K_{RR} &= -Z\sigma_1 \\ K_{LR} &= \mathbb{1}, & K^{RL} &= \mathbb{1} \end{aligned}, \quad (\text{B.85})$$

which implies

$$\mathbb{G} \rightarrow \begin{pmatrix} -\sigma_1 & 0 \\ 0 & Z\sigma_1 \end{pmatrix} \quad (\text{B.86})$$

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